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Integrals

I. Interpretations and properties of definite integrals

A. Alek, you can put your part here if you want.

Section 4.4

II. Fundamental Theorem of Calculus

A. The First Fundamental Theorem of Calculus

If f is a continuous function on [a, b], and F is the indefinite integral of f on [a, b], then $_a\int^b f(x)dx=F(a)-F(b)$
Using the First Fundamental Theorem allows you to obtain integrals over intervals, or the area under a curve.

B. The Second Fundamental Theorem of Calculus

If f is continuous on every open interval containing α, then $d/dx[_\alpha\int^x f(t)dt]=f(x)$

$d/dx[F(t)]_a^x = d/dx[F(x) – F(a)]$ ⟶

C. The Fundamental Theorem of Calculus as analysis of a function.

The Fundamental Theorem can also be used to analyze. A graph can be analyzed by finding the area under its curve on [a, b], which is its definite integral from a to b. By using the Average Value equation, derived from the Fundamental Theorem, you can find the average value on a curve.

$1/(b-a)\ _a\int^b f(x)dx$ = average value of a curve, f(x), on [a,b]

D. Example Problems

1) $_3\int^5 3x^2dx$
2) $_x\int^y f(t)dt$
3) Find the area under the curve y=cos(x)+3on [0,2]
4) Find the average value of $3x^2+1$ on [0,4]

Solutions

1) Since $\int 3x^2=x^3$, then $_3\int^5 5x^2dx=(5)^3-(3)^3$ 125-27=98 Answer: 98
2) F(y)+C-(F(x)-C)=F(y)-F(x) Answer: F(y)-F(x)
3) $_0\int^2 \cos(x) +3=\sin(2)+6-\sin(0)$ Answer: 6

4) $(1/(4-0)\ (x^3+x)]^4_0$ =1/4(64+4) = 68\4 = 17 Answer: 17

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Section

4.2, 4.3, 4.6

III. Numerical Approximations to Definite Integrals

A. Using Riemann Sums

Riemann Sums are a way to estimate the value of a definite integral without using integration.

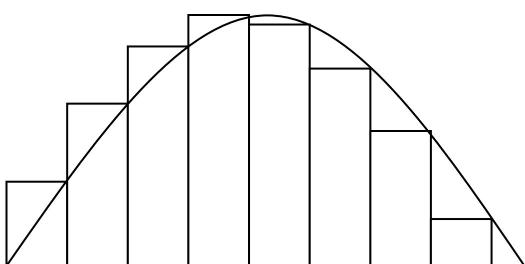
By dividing the area under the curve into rectangles of known width, we can find the area of the rectangles. And by adding those up, we can obtain an estimate of the total area under the curve on the interval.

$$A = \frac{b-a}{n}\sum_{i=1}^{n} f(1+\frac{2i}{n})$$

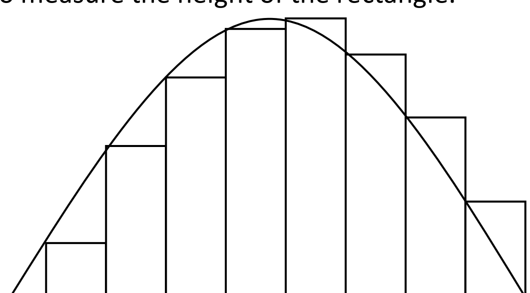
$$base\ of\ rectangles * \sum_{i=1}^{n} height\ of\ rectangles = aproximate\ area$$

Where n is equal to the number of rectangles you have. The height of the rectangles is found by using the function to get a value at that point (if you are only given the function) or by using pre-measured values. The base of the rectangles is whatever you set it up to be based on the length of your interval and your *n* value.

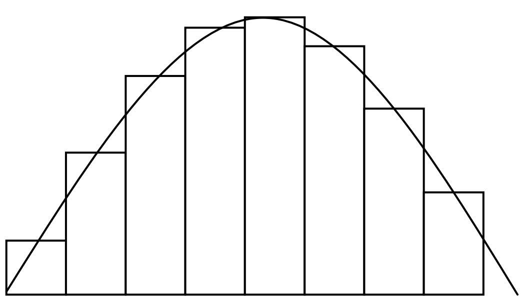
There are three types of Riemann Sums that can be used. There is the Right Hand, Left Hand, and Midpoint. They use different points on the rectangle to measure the height of the rectangle.



Right Hand Riemann Sums – Using Right Riemann Sums, the heights of the right sides of the rectangles are used.



Left Hand Riemann Sums – In taking a Left Hand Sum, the heights of the left hand side of the rectangles are used.



Mid-Point Riemann Sums – The Height of the middle of the rectangles is used.

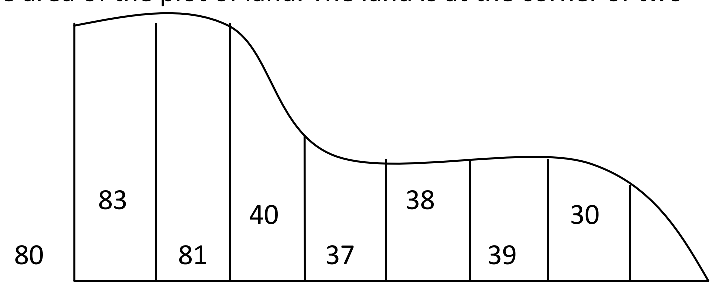
B. Problem Type

Riemann Sum Use

You could easily use Riemann Sums to solve the following problem:

Use Riemann Sums to obtain an estimate of the area of the plot of land. The land is at the corner of two roads and is bounded by a river.

The widths of the sections of land are all 20 ft, and the heights are given in feet.



C. Trapezoid Rule

By using trapezoids instead of rectangles to approximate the area of your region under a curve, you get a better (more accurate) estimation.

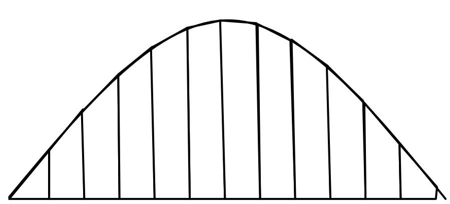
To do this, you need to find the area of trapezoids which requires the heights of both sides of the trapezoids and the width of your segments.

General Term which follows the equation for area of a trapezoid, $A = \frac{1}{2}h(b_1 + b_2)$:

$$A = \frac{b-a}{2n}\,[f(x_0) + 2f(x_1) + 2f(x_2) + . . . + 2f(x_{n-1}) + f(x_n)]$$

Where you are getting the area of the region from *a* to *b*. Again, *n* is the number of trapezoids you have. All the heights are multiplied by 2 (except the first and last) because you need both sides of the trapezoid for each measurement. One height value is used in the calculation of two trapezoids.

An example of a problem using a trapezoid rule could be the same one used above, except using trapezoids instead of Riemann Sums.



Trapezoid Method – As is evident, the use of trapezoids to cover the area under the curve is much more accurate.

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Practice Problems

Fundamental Theorem of Calculus Problems

Questions 1-4 are taken from http://tutorial.math.lamar.edu/Problems/CalcI/CalcI.aspx, a site by a college professor who has put all his calc notes online. I have found it very helpful.

Questions 5-8 are the examples I put on the notes for this section. Questions 9-14 are questions I made up that I hope prove helpful.

1) Differentiate

$$g(x) = \int_{-4}^{x} e^{2t} \cos^2(1-5t)\,dt$$

2) Differentiate

$$\int_{x^2}^{1} \frac{t^4+1}{t^2+1}\,dt$$

3)

a]

$$\int y^2 + y^{-2}\,dy$$

b]

$$\int_1^2 y^2 + y^{-2}\,dy$$

4)

$$\int_{-1}^{2} y^2 + y^{-2}\,dy$$

5) ${}_3\int^5 3x^2dx$

6) ${}_x\int^y f(t)dt$

7) Find the area under the curve y=cos(x)+3on [0,2]

8) Find the average value of $3x^2+1$ on [0,4]

9) ${}_1\int^{10}$ 1/x

10 ${}_4\int^5 x^2+3$

11) ${}_x\int^2$ sin(t)

12)

a] If g(x) = $\int 2x^3$, and g'(x)=f(x), find f(x)

b] Solve for g(x)

13) Find the area under the curve $y=x^3+e^x$ on [2,3]

14)

a] Find the average value of $y=3x^2$ on [1,3]

b] Compare this to the area under $y=3x^2$ on [1,3]

Riemann Sum/Trapezoid Rule Sample Problems

1. Approximate the area under the curve

$f(x) = x^2 + 2, \ -2 \le x \le 1$

with a Riemann sum, using six sub-intervals and right endpoints.

Problem from: Lakeland Schools

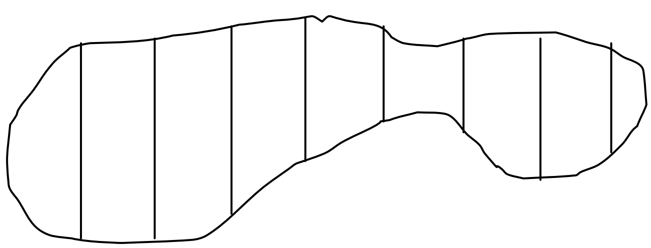
2. Approximate the area under the curve

$$f(x) = \sqrt{x+1}, \qquad -1 \le x \le 0$$

with a Riemann sum, using four sub-intervals and left endpoints.

Problem from: Lakeland Schools

3. Researchers need to find the area of a lake with the following measurements given.



The measurements are all 10 meters from each other and the lengths of the measurements are given in the table below. Don't forget to consider the ends of the lake as zero.

Measurement:	1	2	3	4	5	6	7	8
Length (m)	36	37	35	27	15	16	22	20

Use the Trapezoid method to approximate the area of the lake.

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4. A table below shows values of the rate at which water is flowing out of a pipe. Use a midpoint Riemann sum with four sub intervals to estimate the amount of water flowing out of the pipe from one to nine minutes.

Rate (gal/min)	12	21	35	43	47	44	38	30	19
min	1	2	3	4	5	6	7	8	9

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Solutions to Practice Problems

Fundamental Theorem Problems

1. $g(x)=\int_{-4}^{x} \mathbf{e}^{2t}\cos^2(1-5t)\,dt$

Solution

This is nothing more than an application of the fundamental theorem of calculus

$$g'(x)=\mathbf{e}^{2x}\cos^2(1-5x)$$

2. $\int_{x^2}^{1}\frac{t^4+1}{t^2+1}dt$

Solution

Observe that in order to use the fundamental theorem of calculus, the upper limit must be the variable, and the upper and lower limits must be switched.

$$\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}dt=\frac{d}{dx}\left(-\int_{1}^{x^2}\frac{t^4+1}{t^2+1}dt\right)=-\frac{d}{dx}\int_{1}^{x^2}\frac{t^4+1}{t^2+1}dt$$

The fundamental theorem of calculus requires an x as an upper limit, requiring us to use the chain rule.

$$\frac{d}{dx}\big(g(u)\big)=\frac{d}{du}\big(g(u)\big)\frac{du}{dx}\qquad\text{where } u=f(x)$$

If $u=x^2$

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}dt&=-\frac{d}{dx}\int_{1}^{x^2}\frac{t^4+1}{t^2+1}dt\\&=-\frac{d}{du}\int_{1}^{u}\frac{t^4+1}{t^2+1}dt\ \frac{du}{dx}\qquad\text{where } u=x^2\\&=-\frac{u^4+1}{u^2+1}(2x)\\&=-2x\frac{u^4+1}{u^2+1}\end{aligned}$$

Substitute x back in

$$\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}dt=-2x\frac{\left(x^2\right)^4+1}{\left(x^2\right)^2+1}$$
$$=-2x\frac{x^8+1}{x^4+1}$$

3a. $\int y^2 + y^{-2}\, dy$

Solution

$$\int y^2 + y^{-2}\, dy = \frac{1}{3}y^3 - y^{-1} + c$$

3b. $\int_1^2 y^2 + y^{-2}\, dy$

Solution

$$\int_1^2 y^2 + y^{-2}\, dy = \left(\frac{1}{3}y^3 - \frac{1}{y}\right)\Bigg|_1^2$$
$$=\frac{1}{3}(2)^3 - \frac{1}{2} - \left(\frac{1}{3}(1)^3 - \frac{1}{1}\right)$$
$$=\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1$$
$$=\frac{17}{6}$$

4. $\int_{-1}^2 y^2 + y^{-2}\, dy$

Solution

Since 0 is in the interval, and 0^{-2}, or $1/0^2$ does not exist, this integral is impossible.

5. Since $\int 3x^2 = x^3$, then ${}_3\int^5 5x^2dx = (5)^3 - (3)^3$ 125-27=98 Answer: 98

6. F(y)+C-(F(x)-C)=F(y)-F(x) Answer: F(y)-F(x)

7. ${}_0\int^2 \cos(x) + 3 = \sin(2) + 6 - \sin(0)$ Answer: 6

8. $(1/(4-0)\ (x^3+x)]^4{}_0 = 1/4(64+4) = 68\backslash 4 = 17$ Answer: 17

9. $_1\int^{10}$ 1/x = ln(10) – ln(1)= ln(10) – 0 Answer: ln(10)

10. $_4\int^5 x^2+3 = x^3/3 + 3x\ ]^5_4$ = 125/3 + 15 – 64/3 – 12 Answer: 70/3

Note: 15=45/3 and 12=36/3

11. $_x\int^2$ sin(t) = cos(t) $]^2_x$

$_x\int^2$ sin(t) = -$_2\int^x$ sin(t) = cos(t) $]^x_2$ = cos(x) – cos(2) Answer: cos(x) – cos(2)

Observe that in order to use the fundamental theorem of calculus, the upper limit must be the variable, and the upper and lower limits must be switched.

12. a] If g(x) = $\int 2x^3$, and g'(x)=f(x), find f(x) Answer: f(x) = $2x^3$

Derivatives and integrals are inverse operations

b] Solve for g(x) $(2/(3+1))x^{3+1}$ Answer: $x^4/2$ + C

For integrals not specified, a "+C" is necessary

13. Find the area under the curve $y=x^3+e^x$ on [2,3]

The area under a curve on an interval is the integral of that curve from endpoint to endpoint. $_2\int^3 x^3 + e^x = x^4/4 + e^x\ ]^3_2 = 81/4 + e^3 - 4 - e^4$ Answer: 28.964

14. a] Find the average value of $y=3x^2$ on [1,3] 1/(3-1) $_1\int^3 3x^2$ = ½ $(3x^3/3\ ]^3_1)$ = ½ $(3^3 - 1^3)$ = 26/2 = 13

b] Compare this to the area under $y=3x^2$ on [1,3]

It is exactly half of the area which is 26, since the average value is the area divided by the difference in the interval.

Riemann Sum/Trapezoid Rule Problems

1. $a = -2\ ,\ b = 1\ ,\ \Delta x = \frac{b-a}{n} = \frac{1-(-2)}{6} = \frac{1}{2}\ ,\ f(x) = x^2 + 2$

$$Area \approx (\sum_i f(x_i)\Delta x) = \Delta x\big(f(-1.5) + f(-1) + f(-.5) + f(0) + f(.5) + f(1)\big)$$

$$= \frac{1}{2}((-1.5)^2 + 2 + (-1)^2 + 2 + (-.5)^2 + 2 + 0^2 + 2 + .5^2 + 2 + 1^2 + 2)$$

$$= 8.375$$

2. $a = 1, b = 0\ ,\ \Delta x = \frac{b-a}{n} = \frac{0-(-1)}{4} = \frac{1}{4}\ ,\ f(x) = \sqrt{x+1}$

$$Area \approx \left(\sum_i f(x_i)\,\Delta x\right) = \Delta x\big(f(-1) + f(-.75) + f(-.5) + f(-.25)\big)$$

$$= .25\big(\sqrt{0} + \sqrt{.25} + \sqrt{.5} + \sqrt{.75}\big) = .5183$$

3. For the change in x, we know that the distance between each measurement is 10. We will use this as the height of the trapezoids. The bases are all the other measurements. Using the general formula, we get:

$$Area = \frac{10}{2} * (0 + 2(36) + 2(37) + 2(35) + 2(27) + 2(15) + 2(16) + 2(22) + 2(20) + 0)$$

Remember to multiply all the terms by 2 except for the first one and the last one(which are 0 here). When we calculate the formula above, we get:

$Area \approx 5 * 416 = 2080$m^2

4. Our rectangles must be "two minutes" long in order to cover all the minutes using four sub-intervals. we need the measurements for every other minute (21, 43, 44, 30) since it is a midpoint Riemann sum and we need the measurements that will give us the heights of the middles of the rectangles. Setting it up, we simply get:

$Area \approx \frac{8}{4} * \big(f(2) + f(4) + f(6) + f(8)\big) = 2(21 + 43 + 44 + 30) = 276$ gal of water.